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The extended third-order nonlinear Schrödinger equation and Galilean transformation

V.I. Karpman[a]

Racah Institute of Physics, Hebrew University, Jerusalem 91904, Israel

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Abstract. The extended third-order nonlinear Schrödinger equation and its solutions are studied on the basis of Galilean transformation and generalized Galilean invariance.

PACS. 42.65.Tg Optical solitons; nonlinear guided waves – 52.35.Mw Nonlinear phenomena: waves, wave propagation, and other interactions (including parametric effects, mode coupling, ponderomotive effects, etc.)

1 Introduction

We consider extended third-order cubic nonlinear Schrödinger (ENLS) equation

$$i\partial_T\Psi + \frac{1}{2}\partial_X^2\Psi + |\Psi|^2\Psi + i\alpha_1|\Psi|^2\partial_X\Psi + i\alpha_2\Psi\partial_X|\Psi|^2 + i\alpha_3\partial_X^3\Psi = 0 \quad (1)$$

with real $\alpha_n (n = 1, 2, 3)$. It describes the slow evolution of the wave envelope in nonlinear dispersive systems, taking into account lowest order dispersion corrections, and plays an important role in many areas of nonlinear physics, in particular in nonlinear fiber optics [1,2]. In that context T denotes the distance along the fiber, while X is related to the retarded time. At

$$\alpha_n = 0, \quad (2)$$

equation (1) turns into the regular nonlinear Schrödinger (NLS) equation which is integrable by the inverse scattering method (ISM). The "extra" terms are important for ultra short pulses (e.g., in the femtosecond range). For the particular cases

$$\alpha_1 = 6\alpha_3, \quad \alpha_2 = 0, \quad (3)$$

$$\alpha_1 = 6\alpha_3, \quad \alpha_2 = 3\alpha_3, \quad (4)$$

equation (1) is also integrable. Hirota [3] has found complex N- soliton solutions in case (3) and Sasa and Satsuma [4] have shown the integrability of equation (1) in case (4) by means of the ISM. Not only these integrable cases, but also some other solutions of the ENLS equations with both periodic and vanishing (at infinite X) boundary conditions appear to be rather instructive for the understanding of general phenomena in systems with high order dispersion (HOD). This includes, for example, the question of the integrability conditions of systems with HOD.

It is well known that the regular NLS equation is Galilean invariant and the Galilean transformation (GT) is an important tool for obtaining and interpreting the general solution of NLS equation. The GT can be generalized to HOD systems, in particular to ENLS equation (1). In this paper we demonstrate that the generalized GT permits consideration of many properties and solutions of the ENLS equation from a single point of view. By means of GT, such problems as the derivation of solutions for solitary and periodic waves, integrability condition, self-similar solutions, etc. can be simplified to a great extent. The main purpose of the present work is to give a review of important features of the ENLS equation using the GT.

The paper is organized as follows. In Section 2 we describe the GT and formulate the property of generalized Galilean invariance. The basic solutions of ENLS equation, describing solitary and periodic waves, are derived in Section 3. They are investigated, first, in intrinsic reference frames; then GT leads to corresponding solutions of equation (1). This approach is used also for radiating solitons that play an important role in the evolution of nonlinear pulses. As was shown in reference [5], the radiating soliton can be obtained from a periodic wave by means of the cut-off operation. Performing this in the intrinsic frame of reference we come to a visual picture of the soliton radiation and find its definitive characteristics (wave number, frequency, group velocity, amplitude). In Section 4 we first derive a necessary condition of the integrability of ENLS equation. It consists of the existence of a reference frame where the ENLS equation takes the form of complex modified Korteweg-de Vries (CMKdV)

[a] e-mail: karpman@vms.huji.ac.il

equation. Further reasoning leads to the conclusion that both equations, ENLS and CMKdV, are integrable only in cases considered by Hirota and Sasa-Satsuma. In Section 4.2 it is also demonstrated that the Potasek-Tabor solitons are not the only soliton solutions of the ENLS equation; this is sometimes ignored. The self-similar solutions of ENLS equation are considered in Section 5. The Laurent expansions of self-similar solutions are derived in cases (3) and (4) which are, as it was mentioned, the only integrable cases of ENLS and CMKdV equations.

2 The Galilean transformation

The transformation

$$\Psi(X,T) = \psi(x,t)\exp[i(KX-\Omega T)], \tag{5}$$

where

$$x = X - VT, \quad T = t, \tag{6}$$
$$V = K - 3\alpha_3 K^2, \tag{7}$$
$$\Omega = \frac{1}{2}K^2 - \alpha_3 K^3, \tag{8}$$

transforms equation (1) into

$$i\partial_t\psi + \frac{1}{2}A_2\partial_x^2\psi + q\left|\psi\right|^2\psi + i\alpha_1\left|\psi\right|^2\partial_x\psi + i\alpha_2\psi\partial_x\left|\psi\right|^2 + i\alpha_3\partial_x^3\psi = 0, \tag{9}$$

with

$$A_2 = 1 - 6\alpha_3 K \tag{10}$$

and

$$q = 1 - \alpha_1 K. \tag{11}$$

Equation (5) describes the Galilean transformation (GT). As far as equations (9, 1) have the same form, one can say that the ENLS is Galilean invariant in generalized sense. It is convenient to look for solutions in the intrinsic reference frames starting from equation (9) and then to find the corresponding solution of ENLS equation by means of GT.

3 Solutions of the ENLS equation

3.1 Potasek-Tabor solitons

Let us try to find the solutions of equation (9) which have the form

$$\psi(x,t) = f(x-ut)\exp[-i\varphi(t)], \tag{12}$$

where function $f(y)$ and its derivatives vanish at $|y| \to \infty$. Substituting (12) into (9) we arrive at

$$f = a\,\mathrm{sech}\left[b(x-ut)\right], \tag{13}$$
$$b^2 = \frac{\alpha_1 + 2\alpha_2}{6\alpha_3}a^2 \tag{14}$$
$$u = \alpha_3 b^2, \tag{15}$$
$$\varphi(t) = \frac{1-6\alpha_3 K}{2\alpha_3}ut + \text{const.}, \tag{16}$$
$$K = \frac{\alpha_1 + 2\alpha_2 - 6\alpha_3}{12\alpha_2\alpha_3}. \tag{17}$$

These formulae determine the solution in an intrinsic frame. The corresponding solution of equation (1) is obtained by means of Galilean transformation (5–8) with velocity V expressed through K by equation (7). This gives

$$\Psi_s = a\,\mathrm{sech}\left[b\left(X - X^{(0)} - V_s T\right)\right] \times \exp\left[i\left(\kappa X - \omega T - X^{(1)}\right)\right], \tag{18}$$

where $\kappa = K$ and $V_s = V + u$ is the soliton velocity in the "laboratory" frame and ω the soliton frequency in this frame:

$$V_s = \kappa - 3\alpha_3\kappa^3 + \alpha_3 b^2, \tag{19a}$$
$$\omega = \frac{1}{2}\kappa^2 - \alpha_3\kappa^3 - \frac{1-6\alpha_3\kappa}{2}b^2. \tag{19b}$$

These equations, describing simplest soliton solutions of equation (1), were obtained by Potasek and Tabor [6] by some different approach and will be called the Potasek-Tabor (PT) solitons.

At $\alpha_3 = 0$ the PT solitons do not exist. At $\alpha_2 = 0$ the PT solitons exist only at $\alpha_1 = 6\alpha_3$. In this case κ is arbitrary and

$$a = b. \tag{20}$$

In reference [6] there were also found other families of steady solutions of ENLS equation (e.g., vanishing only at $x \to \infty$, etc.); we shall not discuss them here.

3.2 Solutions of ENLS equation with periodic boundary conditions

Consider the solutions of equation (9) of the form

$$\psi_p(x,t) = \Phi(x)\exp\left[i(1/2)\lambda^2 t\right], \tag{21}$$

where the subscript p means that we impose the periodic boundary condition

$$\Phi\left(x - \frac{\pi}{\Gamma}\right) = \Phi\left(x + \frac{\pi}{\Gamma}\right). \tag{22}$$

Substituting (21) into (9), we have the following equation for the function $\Phi(x)$

$$\frac{1}{2}A_2\partial_x^2\Phi + q\left|\Phi\right|^2\Phi + i\alpha_1\left|\Phi\right|^2\partial_x\Phi + i\alpha_2\Phi\partial_x\left|\Phi\right|^2 + i\alpha_3\partial_x^3\Phi = \frac{1}{2}\lambda^2\Phi. \tag{23}$$

Equations (23, 22) constitute a nonlinear eigenfunction problem with the eigenvalue $\lambda^2/2$. It can be solved numerically by means, e.g., of a kind of shooting method.

At sufficiently small parameter α_3, one can develop an approximate analytical approach considering, first, equation (9) without the third derivative term, i.e.,

$$i\partial_t\tilde{\psi} + \frac{1}{2}A_2\partial_x^2\tilde{\psi} + q\left|\tilde{\psi}\right|^2\tilde{\psi} + i\alpha_1\left|\tilde{\psi}\right|^2\partial_x\tilde{\psi} + i\alpha_2\tilde{\psi}\partial_x\left|\tilde{\psi}\right|^2 = 0. \quad (24)$$

It has soliton solution of the form

$$\tilde{\psi}_s(x,t) = u_s(x)\exp\left[i\phi_s(x) + i(1/2)\lambda^2 t\right], \quad (25a)$$

$$u_s(x) \to 0, \quad |x| \to \infty. \quad (25b)$$

Substituting this into (24) we find

$$u_s(x) = \lambda\sqrt{\frac{2p}{q}}\left[\cosh\left(\frac{2\lambda}{\sqrt{A_2}}x\right) + p\right]^{-1/2}, \quad (26)$$

$$\phi_s(x) = -\frac{\alpha_1 + 2\alpha_2}{2B}\arctan\left[\sqrt{\frac{1-p}{1+p}}\tanh\left(\frac{\lambda x}{\sqrt{A_2}}\right)\right] + \phi_s(0), \quad (27)$$

where

$$p = \sqrt{\frac{A_2}{4B^2\lambda^2 + A_2 q^2}}q, \quad (28)$$

$$B^2 = \frac{4\alpha_1(\alpha_1 + 2\alpha_2) - (\alpha_1 + 2\alpha_2)^2}{12} = \frac{(\alpha_1 + 2\alpha_2)(\alpha_1 - (2/3)\alpha_2)}{4}. \quad (29)$$

The soliton amplitude is expressed by

$$u_0 = \lambda\sqrt{\frac{2p}{(1+p)q}}, \quad (30)$$

and the soliton width is

$$\delta = \sqrt{A_2}/\lambda. \quad (31)$$

Assume, in addition to the smallness of α_3, that

$$\delta \ll 2\pi/\Gamma. \quad (32)$$

Then the full solution of equation (23) inside the period

$$(-\pi/\Gamma, \pi/\Gamma) \quad (33)$$

may be written as

$$\Phi(x) = u_s(x)\exp\left[i\phi_s(x)\right] + f(x), \quad (34)$$

where $f(x)$ is small, near the ends of the period (33). So, from equation (23), we arrive at the following linear equation for f

$$i\alpha_3 f''' + \frac{1}{2}A_2 f'' - \frac{1}{2}\lambda^2 f = 0. \quad (35)$$

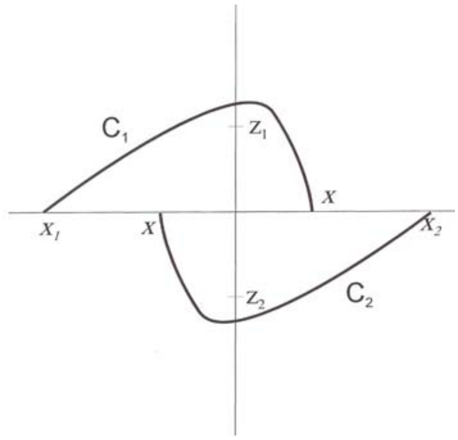

Fig. 1. Contours C_1 and C_2 in the complex plane.

From this it follows that near the ends of the period (33)

$$f(x) \approx \text{const.} \times \exp(ikx), \quad (36)$$

where k is a root of cubic algebraic equation

$$2\alpha_3 k^3 - A_2 k^2 - \lambda^2 = 0. \quad (37)$$

At small α_3, the roots of equation (37) can be written as

$$k = k_{1,2} \approx \pm i\frac{\lambda}{\sqrt{A_2}}, \quad (38)$$

$$k = k_3 \approx A_2/2\alpha_3. \quad (39)$$

The pair of imaginary roots (38) relates to the asymptotic behavior of the soliton pulse, following from (26), while the root (39) represents the wave number in (36). At condition (32), the central pulse is close to the soliton (26) situated on the pedestal wave (cf. Ref. [5], Fig. 1)

$$f(x)\exp\left(i\frac{\lambda^2}{2}t\right). \quad (40)$$

The frequency of the pedestal wave is

$$\omega = -\lambda^2/2 \quad (41a)$$

and the wave number is the real root of equation (37); at small α_3 it is approximately expressed by (39). Equation (37) can be written as

$$\omega(k) = (1/2)A_2k^2 - \alpha_3 k^3, \quad (41b)$$

which is nothing but the dispersion relation for the pedestal wave. The corresponding group velocity $U(k)$ is

$$U(k) = d\omega/dk = A_2 k - 3\alpha_3 k^2. \quad (42a)$$

Using (39) we have

$$U(k) \approx -\frac{A_2^2}{4\alpha_3}. \quad (42b)$$

Therefore

$$\operatorname{sgn} U(k) = -\operatorname{sgn} k = -\operatorname{sgn}\alpha_3. \quad (43)$$

3.3 Soliton-pedestal interaction

It is important that the soliton and the pedestal wave have the same frequency $-\lambda^2/2$. Therefore they *resonantly* interact; this leads to the resonant emission and absorption of the pedestal wave by the soliton. In the steady wave that we consider now, emission and absorption are balanced. The amplitude of pedestal, in a predominant part of period, is equal to const. in (36); it is determined by the strength of soliton-pedestal interaction. The following analysis is based upon a model developed in references [7,8].

The factor depending on x in (25a) can be written as

$$u_s(x)e^{i\phi_s(x)} = u_0 e^{i\phi_s(0)} \exp\left[i\int_0^x dx' Q(x')\right], \tag{44}$$

where

$$Q(x) = -i\frac{u_s'(x)}{u_s(x)} + \phi_s'(x), \quad \phi_s'(x) = -\frac{\alpha_1+2\alpha_2}{2A_2}u_s^2(x). \tag{45}$$

We may call $Q(x)$ the soliton local wave number. In a similar way we define the local wave number of the pedestal

$$\tilde{k}(x) = -i\frac{f'(x)}{f(x)}. \tag{46}$$

If the point x is far enough from the soliton, the function $f(x)$ satisfies equation (35). Then

$$\tilde{k}(x) \approx k. \tag{47}$$

The pedestal-soliton interaction, resulting in the absorption and emission of pedestal wave by the soliton, mathematically displays itself in that $Q(x)$ and $\tilde{k}(x)$ can be regarded as two branches of one analytic function. Near the branch point x_0, where

$$Q(x_0) = \tilde{k}(x_0), \tag{48a}$$

the soliton-pedestal coupling is most intensive. If the root of equation (48a) is real the soliton "sinks" into the pedestal; so their distinction is impossible. It becomes possible only if x_0 is complex and $\mathrm{Im}x_0$ has a proper sign (see below). The coupling decreases with the increase of $|\mathrm{Im}x_0|$. The analytical (asymptotic) description becomes comparatively simple only at sufficiently large $|\mathrm{Im}x_0|$. If equation (48a) has several roots, the main contribution to the coupling comes from the smallest $|\mathrm{Im}x_0|$. At sufficiently large k we can use (47) even in complex plane; then, with asymptotic accuracy,

$$Q(x_0) \approx k. \tag{48b}$$

Introducing

$$x_0 = iz_0 \tag{49}$$

and using (26), (45) we solve equation (48b) to obtain

$$z_0 \approx \pm\frac{\pi\sqrt{A_2}}{4\lambda}\left(1+\frac{2}{\pi}\arcsin p + \frac{2}{\pi}\zeta\right), \tag{50}$$

where ζ is the root of equation

$$\frac{\lambda}{\sqrt{A_2}k}\left(\cos\zeta - \frac{p\sin\zeta}{\sqrt{1-p^2}} + \sqrt{\frac{\alpha_1+2\alpha_2}{\alpha_1-(2/3)\,\alpha_2}}\right) = \sin\zeta - \frac{p(1-\cos\zeta)}{\sqrt{1-p^2}}. \tag{51}$$

Solving (51) at

$$\frac{\sqrt{A_2}\,|k|}{\lambda} \gg 1, \tag{52}$$

we have

$$\zeta = \frac{\lambda}{\sqrt{A_2}k}\left(1+\sqrt{\frac{\alpha_1+2\alpha_2}{\alpha_1-(2/3)\alpha_2}}\right) + O\left(\frac{p}{\sqrt{1-p^2}}\frac{\lambda^2}{A_2k^2}\right). \tag{53a}$$

This expression was obtained if, in addition to (52),

$$\sqrt{1-p^2} \gg \frac{\lambda}{\sqrt{A_2}\,|k|}. \tag{53b}$$

In the opposite case

$$\sqrt{1-p^2} \sim \frac{\lambda}{\sqrt{A_2}\,|k|}, \text{ or } \sqrt{1-p^2} \ll \frac{\lambda}{\sqrt{A_2}\,|k|}, \tag{54a}$$

we take into account the terms $O\left(\zeta^2\right)$ in equation (51). Then we arrive at quadratic equation

$$\gamma\zeta^2 + 2(1+\beta\gamma)\zeta - 2\beta S = 0, \tag{54b}$$

where

$$\beta = \frac{\lambda}{\sqrt{A_2}k}, \quad \gamma = \frac{p}{\sqrt{1-p^2}}, \quad S = 1+\sqrt{\frac{\alpha_1+2\alpha_2}{\alpha_1-(2/3)\alpha_2}}.$$

This gives

$$\zeta = \frac{\sqrt{(1+\beta\gamma)^2+2\beta\gamma S}-(1+\beta\gamma)}{\gamma}. \tag{54c}$$

The chosen root of equation (54b) turns into (53a) at $\gamma \sim 1$.

At small $\sqrt{1-p^2}$ we shall consider only $p \approx +1$. At $p \approx -1$ the soliton-pedestal interaction appears to be strong and this case is not considered. If $q \approx 1$, in addition to $p \approx 1$, the terms with α_1, α_2 can be neglected and equation (9) turns into

$$i\partial_t\psi + \frac{1}{2}A_2\partial_x^2\psi + |\psi|^2\psi + i\alpha_3\partial_x^3\psi = 0. \tag{55}$$

From (28) we have

$$1-p^2 = \frac{4B^2\lambda^2}{4B^2\lambda^2+A_2q^2}. \tag{56a}$$

Thus at $p \approx 1, q \approx 1$

$$2B\lambda \ll 1. \tag{56b}$$

Now let us find the amplitude of pedestal at large distance from the soliton, i.e., const. in equation (36). Consider, first, the case

$$\alpha_3 < 0 \tag{57}$$

and assume that the point x moves along the contour C_1 (Fig. 1) from the point x_1 to the branch point iz_1 and then travels from iz_1 to x (z_1 equal to z_0 with positive sign). This gives

$$u_s(x)\,e^{i\phi_s(x)} + f(x) = u_s(0)e^{i\phi_s(0)} \exp\left[i\int_0^{iz_1} Q(x')dx' + i\int_{iz_1}^{x} \tilde{k}(x')dx' \right], \tag{58}$$

where we put, at the end, $x_1 \to -0$ and took into account that iz_1 is a branch point. Assuming also that $x \gg \delta$, we neglect $u_s(x)$. Then we substitute (45) in (58) and use (47) to obtain

$$f(x) = u_s(iz_0)\exp\left[i\phi_s(iz_0) + i\int_{iz_0}^{0} \tilde{k}(x')\,dx' + ikx \right] \qquad (x \gg \delta). \tag{59}$$

Comparing this with (36) we get

$$\text{const.} = u_s(iz_0)\exp\left[i\phi_s(iz_0) + i\int_{iz_0}^{0} \tilde{k}(x')\,dx' \right] \approx u_s(iz_0)\exp\left[i\phi_s(iz_0) + kz_0 \right]. \tag{60}$$

Here z_0 is taken with the upper sign. From (43) it follows that in case (57)

$$k = -|k|. \tag{61}$$

Therefore

$$\text{const.} \approx u_s(iz_0)e^{i\phi_s(iz_0)} \times \exp\left[-\frac{\pi|k|\sqrt{A_2}}{4\lambda}\left(1 + \frac{2}{\pi}\arcsin p + \zeta\right)\right], \tag{62}$$

where ζ is given by (53a) in case (53b) and by (54c) in case (54a). From (50) and (26), (27) and (30) it follows

$$u_s(iz_0) \approx u_0\frac{(1+p)^{1/2}}{(1-p^2)^{1/4}}(-\zeta)^{-1/2}, \tag{63a}$$

$$\exp\{i\phi_s(iz_0)\} = \exp\left(\frac{\pi}{4}\sqrt{\frac{\alpha_1 + 2\alpha_2}{\alpha_1 - (2/3)\alpha_2}} + i\phi_s(0)\right). \tag{63b}$$

Then at condition (53b) and negative α_3 we have

$$\text{const.} \approx u_0\left(\frac{1+p}{1-p}\right)^{1/4} e^{i\phi_s(0)} \times\left(\frac{\sqrt{A_2}\,|k|}{\lambda}\right)^{1/2}\exp\left[-\frac{\pi\sqrt{A_2}\,|k|}{4\lambda}\left(1 + \frac{2}{\pi}\arcsin p\right) - \frac{\pi}{4}\right]. \tag{64}$$

This expression can also be used for estimations at $\sqrt{1-p^2} \sim \lambda/\sqrt{A_2}\,|k|$. In this case

$$\text{const.} \sim u_0\left(1-p^2\right)^{-1/4} \times e^{i\phi_s(0)}\left(\frac{2\sqrt{A_2}\,|k|}{\lambda}\right)^{1/2}\exp\left\{-\frac{\pi\sqrt{A_2}\,|k|}{2\lambda} - \frac{\pi}{4}\right\}. \tag{65}$$

If

$$\alpha_3 > 0, \tag{66}$$

equation (43) gives $k > 0$, $U(k) < 0$. Now we integrate along the contour C_2 as shown in Figure 1, where iz_2 is the branch point with z_2 is given by equation (50) with the lower sign. Then const. is given by (64), (65) with the replacement $u_0 \to -iu_0$. The above results are in agreement with the estimation of const. in reference [5].

Consider the transition to third-order NLS equation (55). In this case $p \approx 1, q \approx 1$. Then

$$\text{const.} \approx u_0\Lambda\left(\frac{\lambda}{\sqrt{A_2}\,|k|}\right)\left(\frac{\sqrt{A_2}\,|k|}{\lambda}\right)^{1/2}\exp\left(-\frac{\pi\sqrt{A_2}\,|k|}{2\lambda}\right), \tag{67}$$

where $\Lambda(W)$ is a slow function of $W = \lambda/\sqrt{A_2}\,|k|$, which may be of order ten. The explicit calculation of $\Lambda(W)$ is rather tedious and will not be discussed here. We only add that from equation (30) at $p \approx q \approx 1$, it follows $u_0 \approx \lambda$.

These results are obtained in the reference system where the periodic wave is described by (21). If the chain moves with velocity V, the corresponding solutions of equation (1) can be found by means of Galilean transformation

$$\Psi(X,T) = \psi(x,t)\exp\left[i(KX - \Omega T)\right], \tag{68a}$$

where K and Ω are connected with the chain velocity V by equations (7, 8). Using (21), we write

$$\Psi_p(X,T) = \Phi(x)\,e^{iKx}\exp\left\{i\left[\frac{1}{2}\lambda^2 + (KV - \Omega)\,t\right]\right\}, \tag{68b}$$

where $\Phi(x)$ satisfies equation (23). Finding $K(V)$ from (7), we have

$$K = \frac{1 - \sqrt{1 - 12\alpha_3 V}}{6\alpha_3}. \tag{69a}$$

Respectively

$$A_2 = \sqrt{1 - 12\alpha_3 V}. \tag{69b}$$

Denoting

$$\chi(x) = \Phi(x)\,e^{iKx}, \tag{70}$$

we write

$$\Psi_p = \chi(x)\,e^{i\Lambda t} \quad (x = X - VT), \tag{71}$$

where

$$\Lambda = \frac{1}{2}\lambda^2 + KV - \Omega = \frac{1}{2}\lambda^2 + \frac{1}{2}K^2 - 2\alpha_3 K^3. \tag{72}$$

From (49, 51) and (23) we find the the following equation for χ

$$-iV\partial_x\chi+\frac{1}{2}\partial_x^2\chi+|\chi|^2\chi+i\alpha_1|\chi|^2\partial_x\chi+i\alpha_2\chi\partial_x|\chi|^2+i\alpha_3\partial_x^3\chi=\Lambda\chi. \quad (73)$$

Of course, this equation also emerges by substituting (71) into (1).

According to (22), $\Phi(x)$ is a periodic function with the period $2\pi/\Gamma$. Assume that

$$\Gamma=K/n,$$

where n is integer number. Then $\chi(x)$ is also a periodic function and

$$\chi(x-\pi/\Gamma)=\chi(x+\pi/\Gamma). \quad (74)$$

Near the ends of the interval of periodicity

$$\chi(x)\approx \text{const.}\times e^{i\kappa x}, \quad (75)$$

where const. is small.Substituting (75) into linearized equation (73) we arrive at equation for κ

$$\alpha_3\kappa^3-(1/2)\kappa^2-(\Lambda-\kappa V)=0. \quad (76)$$

Using (72, 37, 10) and (7), we rewrite this as

$$\alpha_3\left[\kappa^3-(k+K)^3-3(\kappa-k-K)K^2\right]-\frac{1}{2}\left[(\kappa-K)^2-k^2\right]=0.$$

This can be regarded as a cubic equation for $\kappa-K$; it has only one real root $\kappa-K=k$. Therefore

$$\kappa=k+K. \quad (77)$$

The nonlinear eigenfunction problem (73) with boundary condition (75) was numerically solved in reference [5] and it was demonstrated that numerical results are in agreement with the above conclusions.

3.4 Resonantly radiating solitons

Define a cutoff operation, transforming $\Phi(x)$ into a single pulse [5]:

$$\Phi_c(x)=\Phi(x)\rho(x), \quad (78)$$

where $\rho(x)$ is a cutoff factor

$$\rho(x)=\frac{1}{2}\left[\tanh\left(\frac{x-x_0+\Delta x}{\gamma}\right)\right]-\left[\tanh\left(\frac{x-x_0-\Delta x}{\gamma}\right)\right]. \quad (79)$$

Here x_0 is the center of the pulse and Δx is the width of $\Phi_c(x)$. According to (58), $\rho(x)$ vanishes at $|x-x_0|\to\infty$ and the parameter γ characterizes the sharpness of vanishing. Assuming that γ is small enough and Δx is such that the factor $\rho(x)$ cuts off only the wings without essentially disturbing the pulse in $\Phi(x)$ (they can be properly chosen by numerical tests) we then take $\Phi_c(x)$ as the initial condition to equation (9)

$$\psi(x,0)=\Phi_c(x). \quad (80)$$

So, initially we have an isolated soliton-like pulse. From the numerical solution [5] we find, neglecting small transient effects, that the pulse (80) asymptotically develops into

$$\psi(x,t)\approx u_s(x)\exp\left[i\phi_s(x)+i\frac{\lambda^2}{2}t\right]+f(x)\exp\left(i\frac{\lambda^2}{2}t\right)\Theta(Ux)\,\Theta(|U|t-|x|), \quad (81)$$

where $\Theta(X)$ is the Heaviside function and $U=U(k)$ is the pedestal group velocity (42a). This result can be explained as follows. The soliton, described by the first term in (81), resonantly interacts with the wave of continuous spectrum with frequency $\omega(k)=-\lambda^2/2$. It can be, therefore, resonantly emitted and absorbed by the soliton. However, the incident wave is now absent. So the soliton emits the resonant wave at the expense of its own energy and momentum. The second term in (81) describes such a wave. It is a wave train consisting of the piece of pedestal wave (40) that is radiated by the soliton in the direction of group velocity U and has length $|U|t$. At condition (52), the coupling between the soliton and radiated wave is small.

The radiation causes an attenuation of the soliton. However, due to the smallness of radiation we may neglect the soliton damping during rather large time intervals of radiation. This can be called *adiabatic* approximation and expression (81) is written in this approximation. The amplitude of $f(x)$ in the adiabatic approximation is given by (64) and (65). All these conclusions were confirmed by numerical modeling [5]. The numerical experiments, performed in this work directly for equation (1), show that the radiating solitons (in fact, quasisolitons) play an important role in the long time evolution of waves described by the ENLS equation.

Considering $p\approx 1$, $q\approx 1$ we arrive at the radiating solitons described by the third-order NLS equation (55). In this case the amplitude of radiation is expressed by (67). It is consistent with earlier works [7–12] on the third-order NLS equation.

The phenomenon of soliton radiation results in the *irreversible* decay of solitons; therefore it takes place only if the ENLS equation is not completely integrable. We now return to the integrable ENLS equations (which are, in fact, exceptional) and consider in more detail the soliton properties in these cases.

4 Integrability conditions of the ENLS equation

4.1 Necessary condition of the integrability

The integrability conditions for the ENLS equation follow from the Painlevé conjecture. The Painlevé analysis of equation (1) at $\alpha_2 \neq 0$ [13] shows that it is integrable only in case (4). The ISM for this case have been developed, as it was mentioned above, by Sasa and Satsuma [4]. Making a similar Painlevé analysis of equation (1) at $\alpha_2 = 0$, one arrives at integrability condition (3). Thus the *necessary* condition of integrability is

$$\alpha_1 = 6\alpha_3. \tag{82}$$

Perform the Galilean transformation (5–7) with

$$K = 1/\alpha_1. \tag{83a}$$

Then from (10, 11) it follows

$$A_2 = 0, \quad q = 0 \tag{83b}$$

and equation (9) takes the form

$$\partial_t \psi + 6\alpha_3 |\psi|^2 \partial_x \psi + \alpha_2 \psi \partial_x |\psi|^2 + \alpha_3 \partial_x^3 \psi = 0 \tag{84}$$

that can be viewed as complex modified Korteweg-de Vries (CMKdV) equation. One can easily prove that any particular CMKdV equation can be transformed to the form (84) by an appropriate transformation of independent and dependent variables, i.e., equation (84) can be considered as the general form of CMKdV equation. We therefore conclude that

A necessary condition of the complete integrability of ENLS equation is that there exists a reference frame, where equation (1) takes the form of CMKdV equation.

On the other hand, the Painlevé test of equation (84) shows that it is completely integrable only in two cases: $\alpha_2 = 3\alpha_3$ and $\alpha_2 = 0$. In the first of them, the inverse scattering transform has been developed by Sasa and Satsuma [4]. In the second case, the $L{-}A$ pair can be easily found by the proper generalization of Wadati theory [14], elaborated for the real MKdV equation. Therefore both equations (84) and (1) are completely integrable if $\alpha_1 = 6\alpha_3$ and, in addition, $\alpha_2 = 3\alpha_3$ or $\alpha_2 = 0$.

4.2 Solitons of the integrable ENLS equation

In some cases, the general form of soliton solutions is rather difficult to find even for integrable equations. Then it may be helpful to employ an approach based on the Gelfand-Levitan-Marchenko (GLM) equation. This can be demonstrated for Sasa-Satsuma case (4), when the ENLS equation has the form

$$i\partial_T \Psi + \frac{1}{2}\partial_X^2 \Psi + |\Psi|^2 \Psi + i\varepsilon\left(\partial_X^3 \Psi + 6|\Psi|^2 \partial_X \Psi + 3\Psi \partial_X |\Psi|^2\right) = 0, \tag{85}$$

where $\varepsilon = \alpha_3$. According to reference [4], the one-soliton solution of GLM equation, associated with equation (85), is

$$\Psi(X,T) = \frac{2\eta |C| e^{iB}\left(e^{A} + Ce^{-A}\right)}{e^{2A} + |C|^2 e^{-2A} + 2|C|^2}. \tag{86}$$

Here

$$A = \eta\left\{X - \left[\xi + \varepsilon\left(\eta^2 - 3\xi^2\right)\right]T - X^{(0)}\right\}, \tag{87a}$$

$$B = \xi\left\{X + \left[\frac{\eta^2 - \xi^2}{2\xi} + \varepsilon\left(\xi^2 - 3\eta^2\right)\right]T - X^{(1)}\right\}, \tag{87b}$$

$$C = 1 - \frac{i\eta}{\xi - 1/6\varepsilon}. \tag{88}$$

Now consider two limit cases.

(i): $\varepsilon \ll 1, \quad \xi \ll 1/6\varepsilon, \eta \ll 1/6\varepsilon$.

Then

$$|C| \approx 1, \quad \arg C \approx -6\varepsilon. \tag{89}$$

and expression (86) is reduced to

$$\Psi(X,T) \approx \eta \exp\left[i\xi\left(X - \frac{\eta^2 - \xi^2}{2\xi}T - X^{(1)}\right)\right] \times \mathrm{sech}\left[\eta\left(X - \xi T - X^{(0)}\right)\right]. \tag{90}$$

It describes the NLS soliton with amplitude η and velocity ξ. One can see that the maximum of $|\Psi|$ in equation (86) becomes flatter with the increase of $|C|$. At $|C| > 2$, there are two maximums of $|\Psi|$.

(ii): $\varepsilon \ll 1$, $6\varepsilon\left(\xi - \frac{1}{6\varepsilon}\right) \ll 1$.

Now we can simplify (86) assuming $|C| \to \infty$. Then

$$\Psi(X,T) \approx \left(1\big/\sqrt{2}\right) i\eta e^{iB}\, \mathrm{sech}\left(A + \ln\sqrt{2}\right), \tag{91}$$

where A and B are given by (87a, b) with $\xi = 1/6\varepsilon$, i.e.,

$$A \approx \eta\left[X - X^{(0)} - \left(\varepsilon\eta^2 + \frac{1}{12\varepsilon}\right)T\right], \tag{92a}$$

$$B \approx \xi\left[X - X^{(1)} - \left(\frac{8\varepsilon\eta^2}{3} + \frac{1}{18\varepsilon}\right)T\right]. \tag{92b}$$

Comparing this with (18) and (19a, b) we see that expressions (91–92a, b) represent a PT soliton. Unlike (90), equations (91–92a, b) do not approach the NLS soliton at $\varepsilon \to 0$. This is because at large C the third-order dispersion term in equation (1) essentially competes also with the nonlinear terms containing α_1, α_2.

Now let us consider the Hirota conditions (3), introduced in reference [3]. Then equation (84) takes the form

$$\partial_t \psi + 6\varepsilon |\psi|^2 \partial_x \psi + \varepsilon \partial_x^3 \psi = 0. \tag{93}$$

The one-soliton solution of this equation can be easily found; it reads

$$\psi_s(x,t) = a\, \mathrm{sech}\left[a(x - ct)\right] \exp\left[i\left(px - \sigma t + \vartheta\right)\right], \tag{94}$$

where

$$c = -3\varepsilon p^2 + \varepsilon a^2, \tag{95a}$$

$$\sigma = -\varepsilon p^3 + 3\varepsilon p a^2, \tag{95b}$$

with arbitrary parameters a, p and ϑ. At $p = \vartheta = 0$ we have solitons of the real MKdV equation. Hirota [3] has found N-soliton solutions composed at $x \to \pm\infty$ of complex solitons (94). At $p = \vartheta = 0$ they turn into the N-soliton solutions of real MKdV equation derived by Wadati [14]. Performing the Galilean transformation of equation (94), we arrive at the PT solitons at $\alpha_1 = 6\alpha_3 = 6\varepsilon$ expressed by equations (18–20). Thus at Hirota conditions, the solitary waves are nothing more than PT solitons. As it was mentioned above, at $\alpha_1 = 6\alpha_3 = 6\varepsilon$ there exists $L - A$ pair that is clearly a modification of the pair found by Wadati [14].

5 Self-similar solutions

We restrict ourselves to the integrable cases. Then we can start with the CMKdV equation (84) and consider cases (3) and (4). Equation (84) is, in fact, the system of equations for ψ and its complex conjugate. Replacing

$$\psi \Rightarrow R, \quad \psi^* \Rightarrow S, \tag{96a}$$

and using notations

$$\varepsilon = \alpha_3, \quad \mu = \alpha_2, \tag{96b}$$

we arrive at the system of equations

$$\partial_t R + (6\varepsilon + \mu) RS\partial_x R + \mu R^2 \partial_x S + \varepsilon \partial_x^3 R = 0, \tag{97a}$$

$$\partial_t S + (6\varepsilon + \mu) RS\partial_x S + \mu S^2 \partial_x R + \varepsilon \partial_x^3 S = 0. \tag{97b}$$

This system admits a self-similar Ansatz

$$R(x,t) = (3t)^{-1/3} r(z), \tag{98a}$$

$$S(x,t) = (3t)^{-1/3} s(z), \tag{98b}$$

$$z = (3t)^{-1/3} x. \tag{99}$$

Substituting (98a, b) and (99) into (97a, b) we have

$$\varepsilon r''' + (6\varepsilon + \mu) srr' + \mu r^2 s' - (zr)' = 0, \tag{100a}$$

$$\varepsilon s''' + (6\varepsilon + \mu) rss' + \mu s^2 r' - (zs)' = 0, \tag{100b}$$

where the prime means derivative over z.

Equations (100a, b) have a particular solution $r = s$, corresponding to the real MKdV equation that follows from (84) at real initial conditions. Then r satisfies equation

$$\varepsilon r''' + 2(3\varepsilon + \mu) r^2 r' - (zr)' = 0. \tag{101}$$

After one integration, we have

$$\varepsilon r'' + \frac{2}{3}(3\varepsilon + \mu) r^3 - zr = \text{const.} \tag{102}$$

This is the second Painlevé equation for the second Painlevé transcendent P_{II}. All its movable singularities are poles [15], in agreement with the complete integrability of the real MKdV equation.

We now look for the solution of equations (100a, b) in the form of Laurent series

$$r(z) = a_0 + \sum_{n=1}^{\infty}\left[\frac{A_n}{(z-z_0)^n} + a_n (z-z_0)^n\right], \tag{103a}$$

$$s(z) = b_0 + \sum_{n=1}^{\infty}\left[\frac{B_n}{(z-z_0)^n} + b_n (z-z_0)^n\right]. \tag{103b}$$

As far as we consider an integrable variety of equation (84), the system (100a, b) should be also integrable. Then, according to the Painlevé conjecture, the movable singularities in (103a, b) can only be poles; so the number of terms containing $(z - z_0)^{-n}$ must be finite. Then, substituting (103a, b) into (100a, b) we find

$$A_1 B_1 = -\frac{3\varepsilon}{3\varepsilon + \mu}, \tag{104}$$

$$A_n = B_n = 0 \quad (n > 1). \tag{105}$$

For definiteness, we assume

$$\varepsilon > 0, \quad 3\varepsilon + \mu > 0. \tag{106}$$

Then we write

$$r(z) = \frac{A_1}{z - z_0} + \sum_{n=0}^{\infty} a_n (z-z_0)^n, \tag{107a}$$

$$s(z) = \frac{B_1}{z - z_0} + \sum_{n=0}^{\infty} b_n (z-z_0)^n. \tag{107b}$$

Deriving equations for a_n, b_n, we have at $n = 0, 1$

$$(6\varepsilon + 3\mu) B_1 a_0 + (6\varepsilon + \mu) A_1 b_0 = 0, \tag{108a}$$

$$(6\varepsilon + \mu) B_1 a_0 + (6\varepsilon + 3\mu) A_1 b_0 = 0, \tag{108b}$$

$$2\mu B_1 a_1 + 6\varepsilon A_1 b_1 = z_0 - F_1/A_1, \tag{109a}$$

$$6\varepsilon B_1 a_1 + 2\mu A_1 b_1 = z_0 - G_1/B_1, \tag{109b}$$

where

$$F_1 = (6\varepsilon A_1 b_0 + \mu B_1 a_0) a_0, \tag{110a}$$

$$G_1 = (6\varepsilon B_1 a_0 + \mu A_1 b_0) b_0. \tag{110b}$$

For $n \geq 2$, we write

$$X(n) B_1 a_n + Y(n) A_1 b_n = B_1 \mathrm{K}(n), \tag{111a}$$

$$Y(n) B_1 a_n + X(n) A_1 b_n = A_1 \Lambda(n). \tag{111b}$$

Then

$$X(n) = \varepsilon\Big[3\varepsilon(n-1)\left(n^2 - 2n - 6\right) + \mu\left(n^3 - 3n^2 - n + 9\right)\Big], \tag{112a}$$

$$Y(n) = 3\varepsilon\left[6\varepsilon - \mu(n-1)\right], \tag{112b}$$

$$\mathrm{K}(n) = (3\varepsilon + \mu)\left[(n-2)(a_{n-3} + z_0 a_{n-2}) + F_n\right], \tag{113a}$$

$$\Lambda(n) = (3\varepsilon + \mu)\left[(n-2)(b_{n-3} + z_0 b_{n-2}) + G_n\right], \tag{113b}$$

and F_n, G_n are third-order nonlinear forms composed of A_1, B_1 and a_m, b_m with $m \le n-1$. In particular,

$$F_2 = (6\varepsilon - \mu)(A_1 b_1 - B_1 a_1) a_0, \tag{114}$$

$$F_3 = (6\varepsilon + \mu)\Big[A_1 (a_0 b_2 + b_0 a_2) - 2(A_1 b_0 + B_1 a_0) a_2 - (B_1 a_1 + a_0 b_0) a_1\Big] - \mu\Big(4A_1 a_0 b_2 - 2B_1 a_0 a_2 + a_0^2 b_1 + 2A_1 a_1 b_1 - B_1 a_1^2\Big). \tag{115}$$

Expressions for G_n can be obtained by the rule

$$A_1 \Leftrightarrow B_1, \quad a_m \Leftrightarrow b_m. \tag{116}$$

In fact, this rule is applicable also to (111a, b–113a, b).

Solving (111a, b), we have

$$B_1 a_n = \frac{B_1 \mathrm{X}(n) K(n) - A_1 Y(n) \Lambda(n)}{\mathrm{X}^2(n) - Y^2(n)}, \quad n \ge 2. \tag{117}$$

A similar expression for $A_1 b_n$ follows from (116). Defining

$$\Delta = \frac{3\varepsilon^2}{3\varepsilon + \mu}\left[\mathrm{X}^2(n) - Y^2(n)\right], \tag{118}$$

we first investigate the roots of equation

$$\Delta = 0. \tag{119}$$

This can be easily done by means of expressions

$$\mathrm{X}(n) + Y(n) = (3\varepsilon + \mu)\varepsilon(n-3)\left(n^2 - 4\right), \tag{120}$$

$$\mathrm{X}(n) - Y(n) = \varepsilon\Big[3\varepsilon n(n+1)(n-4) + \mu\left(n^3 - 3n^2 + 2n + 6\right)\Big]. \tag{121}$$

In particular, for the integrable cases

$$\mathrm{X}(n) - Y(n) = 3\varepsilon^2 n(n+1)(n-4) \quad (\mu = 0), \tag{122}$$

$$\mathrm{X}(n) - Y(n) = 6\varepsilon^2\left(n^2 - 1\right)(n-3) \quad (\mu = 3\varepsilon). \tag{123}$$

Respectively, equation (119) has roots $n = 2, 3, 4$ at $\mu = 0$, and $n = 1, 2, 3$ at $\mu = 3\varepsilon$. In the first case equation (84) turns into the Hirota CMKdV equation, while in the second case we arrive at the Sasa-Satsuma CMKdV equation

$$\partial_t \psi + 6\varepsilon |\psi|^2 \partial_x \psi + \mu \psi \partial_x |\psi|^2 + \varepsilon \partial_x^3 \psi = 0. \tag{124}$$

(i) Let us first investigate the Laurent series (103a, b) for the solution of the Hirota equation. From (104) it follows

$$A_1 B_1 = -1. \tag{125}$$

Equations (108a, b) and (109a, b) give

$$B_1 a_0 + A_1 b_0 = 0, \tag{126}$$

$$6\varepsilon B_1 a_1 = z_0 - 6\varepsilon a_0 b_0, \tag{127a}$$

$$6\varepsilon A_1 b_1 = z_0 - 6\varepsilon a_0 b_0. \tag{127b}$$

Therefore

$$B_1 a_1 - A_1 b_1 = 0. \tag{128}$$

Using these equations we have

$$F_1 = -G_1 = 6\varepsilon A_1 b_0 a_0, \tag{129a}$$

$$F_2 = G_2 = 0. \tag{129b}$$

Equations (111a, b) at $n = 2$ give only one equation

$$B_1 a_2 - A_1 b_2 = 0 \tag{130}$$

(the second equation coincides with (109) because equation (98) has a root $n = 2$). Equations (111a, b) at $n = 3$ are reduced to

$$6\varepsilon(B_1 a_3 - A_1 b_3) = -B_1(a_0 + a_1 z_0 + F_3), \tag{131a}$$

$$6\varepsilon(B_1 a_3 - A_1 b_3) = A_1(b_0 + b_1 z_0 + G_3). \tag{131b}$$

This system is also degenerated (because equation (119) has a root $n = 3$). The condition of compatibility of equations (131a, 131b) is

$$B_1(a_0 + F_3) + A_1(b_0 + G_3) + z_0(B_1 a_1 + A_1 b_1) = 0. \tag{132a}$$

Using (115), (126–128) and (130), we get a simplified expression for F_3

$$F_3 = -6\varepsilon a_1(B_1 a_1 + a_0 b_0) = -z_0 a_1. \tag{132b}$$

Substituting (132b) into (132a), we find that the compatibility condition is fulfilled and system (131a, b) can be reduced to a single equation

$$12\varepsilon(A_1 b_3 - B_1 a_3) = B_1 a_0 - A_1 b_0. \tag{133}$$

Now consider $n = 4$. Then equation (119) is again satisfied and so system (111a, b) is degenerated: it looks as

$$6\varepsilon(B_1 a_4 + A_1 b_4) = B_1[2(a_1 + z_0 a_2) + F_4], \tag{134a}$$

$$6\varepsilon(B_1 a_4 + A_1 b_4) = A_1[2(b_1 + z_0 b_2) + G_4]. \tag{134b}$$

Calculating F_4 at $\mu = 0$, we have

$$B_1 F_4 = 6\varepsilon\left[a_1 b_2 - (a_0 b_3 + b_0 a_3) - B_1 a_1(a_1 b_0 + b_1 a_0) - 3B_1 a_3(B_1 a_0 + A_1 b_0)\right] - 2z_0 B_1 a_2. \tag{135}$$

Taking into account (128) and (130), we can write the solvability condition of system (134a, b) as

$$B_1 F_4 = A_1 G_4. \tag{136}$$

This is satisfied due to (116). Then system (134a, b) is reduced to one equation, e.g. (134a).

Thus equations (111a, b) determine a_n, b_n $(n \le 4)$ not uniquely: the solutions contain four arbitrary quantities. One more arbitrariness is contained in the solution of equation (125) and, finally, z_0 is also arbitrary. As for equations (111a, b) at $n > 4$, we see from (118), (120) and (122) that they uniquely determine a_n, b_n through A_1, B_1 and a_m, $b_m (m < n)$. From that it follows that expressions (108a, b) contain six arbitrary constants and therefore (108a, b) represent the general solution of ODE system (100a, b) that must contain six arbitrary constants of integration.

(ii) Consider now the case $\mu = 3\varepsilon$, when equation (84) turns into the Sasa-Satsuma equation, the second integrable CMKdV equation. Its solution can be written in the form (107a, b) as well. Now the coefficients of these Laurent series are determined by equations

$$A_1 B_1 = -1/2 \tag{137}$$

[following from (104)] and

$$a_0 = b_0 = 0, \tag{138}$$
$$B_1 a_1 - A_1 b_1 = C_1, \tag{139}$$
$$A_1 b_2 = B_1 a_2 = C_2, \tag{140}$$
$$B_1 a_3 + A_1 b_3 = C_3, \tag{141}$$
$$B_1 a_3 - A_1 b_3 = C_4. \tag{142}$$

Here $C_n (n = 1, 2, 3, 4)$ are arbitrary parameters. The determinant of system (111a, b) is

$$\Delta = 6\varepsilon^2 \left(n^2 - 1\right) \left(n^2 - 4\right) (n - 3)^2 . \tag{143}$$

So, now the systems of equations for a_n, b_n with $n > 3$ are not degenerated. In additions to C_n, we have two more arbitrary parameters: one follows from (137) and the second is z_0. We therefore conclude that the coefficients in (107a, b) depend on six arbitrary parameters, corresponding to six arbitrary constants of integration in the general solution of ODE system (100a, b). Thus expressions (107a, b) indeed represent the general solution of that system. In both cases (i) and (ii) the general solution has only movable poles as singularities, in agreement with the Painlevé conjecture.

Using the Galilean transformation (5), we obtain the corresponding solutions of ENLS equation (1) with the coefficients satisfying conditions (3), (4). It is not difficult to see that there are no other self-similar solutions of equation (84).

6 Conclusions

We have considered, from a single point of view based on the systematic use of Galilean transformation (GT), important solutions of the ENLS equation (Potasek-Tabor solitons, nonlinear periodic waves, radiating solitons, etc.). The GT, being applied to equation (1), transforms it to equation (9) which is solved in the intrinsic frame of a wave. After appropriate Galilean transformation we then arrive at the solution of equation (1) describing the wave in question moving with velocity V. This approach leads also to a necessary condition of the integrability of ENLS equation that reads: there should exist a GT transforming equation (1) to a complex modified Korteweg-de Vries (CMKdV) equation. The Painlevé test of CMKdV equation then shows that it is integrable only at Hirota and Sasa-Satsuma conditions [3,4]. So the ENLS equation is integrable only in these cases (Sect. 4). The self-similar solutions of CMKdV equation confirm this conclusion (Sect. 5).

The GT transformation can be readily used also for the 4th order ENLS equation [16]. More general extensions will be considered elsewhere.

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